EARTHQUAKE ENGINEERING AND STRUCTURAL DYNAMICS
Earthquake Engng. Struct. Dyn. **27**, 957–976 (1998)

AN LMI APPROACH TO H_∞-CONTROL OF TIME-DELAY SYSTEMS FOR THE BENCHMARK PROBLEM

MAGDI S. MAHMOUD[1,†], MOHAMAD J. TERRO[2,*,‡] AND MOHAMED ABDEL-ROHMAN[2,†]

[1] *Department of Electrical and Computer Engineering, Kuwait University, PO Box 5969, Safat-13060, Kuwait*
[2] *Department of Civil Engineering, Kuwait University, PO Box 5969, Safat-13060, Kuwait*

SUMMARY

In this paper, three H_∞-control design methods are developed and applied to a three-storey building with an active mass damper as a control mechanism. The system of equations of the structural system, including the actuator and sensors, has been developed directly from experimentally derived data which forms the basis of the benchmark study discussed in this paper. The building plus the damper are modelled as a nominally linear system with input as well as state delays. Feedback control synthesis are first performed by using either of the two forms; the first is a pure state feedback and the other is a static output feedback. The analytical results are cast into a Linear Matrix Inequality (LMI) framework which can be solved numerically by efficient interior-point methods. The developed system is subjected to two historical earthquake excitation inputs (ElCentro and Hachinohe) and to the Kanai–Tajimi filter. The response is given in the form of indices in order to compare with other solutions of the benchmark problem. In addition, simulation results pertinent to the developed techniques are presented. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: linear matrix inequality; time-delay; active mass damper; control mechanism

1. INTRODUCTION

The last two decades have witnessed extensive research in the field of active structural control which is a relatively new technology.[1–3] Various control methods have been introduced as a means to guarantee safety for modern civil structures exposed to severe dynamic excitations such as those experienced due to wind or earthquakes. The time has come to attempt to compare and validate those control methods and choose the best-possible technique that fits structural civil engineering applications.

In the benchmark problem, which is the subject of this research, experimental data obtained from a model representing a three storey building is used to derive an evaluation model presented in the problem definition of the structural system. The experimental setup on which the benchmark problem is used was achieved at the Structural Dynamics and Control/Engineering Laboratory (SDC/EEL) at the University of Notre Dame.[4] The developed evaluation model is considered as the real structural system since it accurately describes the real behavior of the structure and the actuators/sensors observed in the laboratory. A number of evaluation criteria and control design constraints are provided with the benchmark study to assess if the developed controllers perform similarly in the laboratory settings.

In control engineering systems design, the primary objective is to construct systems with better performance rates. The H_∞-norm of the closed-loop system has been often considered an important performance

* Correspondence to: Mohamad J. Terro, Department of Civil Engineering, Kuwait University, PO Box 5969, Safat-13060, Kuwait. E-mail: Terro@kuc01.kuniv.edu.kw
† Professor
‡ Assistant Professor

CCC 0098–8847/98/090957–20$17·50

Received 15 May 1997
Revised 2 October 1997

index.[7,8] An integral part of the dynamical behaviour of different physical processes is the existence of inherent time-delays.[9] It turns out that time delay has been one of the major sources of instability in control systems.[10] Stability problems of time-delay systems have therefore been the subject of numerous studies.[9,11] For systems without delay, major results based on H_∞-control design can be found References 7, 8, 12, 13 and their bibliographies. Except for the classical stability results of References 14–17, stabilization and control design problems of a wide class of time-delay systems with delays in both the state and input has not been fully examined in the literature.

In this paper, H_∞-feedback control synthesis methods are developed for the control of vibrations in structures as modelled by nominally linear systems with input as well as state delays. The developed results are then cast into Linear Matrix Inequality (LMI) forms which are easily computable for checking the stabilization of the controlled systems. These synthesis methods are applied to the structural model exposed to vibrations due to two historical earthquake excitation inputs (El-Centro and Hachinohe) and Kanai–Tajimi filter. The advantages of our technique are the guaranteed closed-loop stabilization, in addition to the generality of the design procedure and the flexibility and systematic computation of the gain matrices.

In the sequel, we use W^t, W^{-1}, $\lambda(W)$ to denote, respectively, the transpose of, the inverse of and the eigenvalues of any square matrix W. The vector norm is taken to be Euclidean and the matrix norm is the corresponding induced one; that is $\|W\| = \lambda_M^{1/2}(W^t, W)$; where $\lambda_{M(m)}(W)$ stands for the operation of taking the maximum (minimum) eigenvalue of W. We use $W > 0 (W < 0)$ to denote a positive- (negative-) definite matrix W. Let I_s be the unit matrix of order s and $e_s = [1\,1\,1\,1 \ldots 1]^t$. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

2. EXPERIMENTAL MODEL

The evaluation model is based on an actively controlled three-storey, single bay, model building originally considered by Dyke *et al.*,[5,6] (see Figure 1). The building frame is made of steel with a total height and weight of 158 cm and 227 kg, respectively. The weight of the structure is distributed evenly amongst the three floors. The first three modes of the model structural system are at 5·81, 17·68 and 28·53 Hz, with associated damping ratios of 0·33, 0·23 and 0·30 per cent, respectively. A moving Active Mass Driver (AMD), weighing 5·2 kg, was used in this experiment. The total mass of the structure with the AMD reached 309 kg.

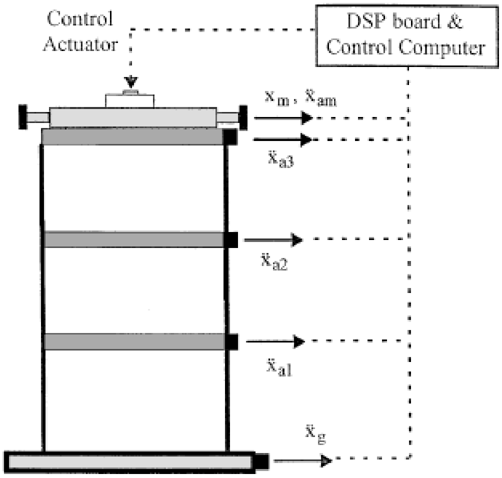

Figure 1. Schematic diagram of the experimental setup

3. EVALUATION MODEL

The uncontrolled system representing the model experimental setup is governed by the following differential state equations:

$$\dot{x} = Ax + Bu + E\ddot{x}_g \tag{1}$$

$$y = C_y x + D_y u + F_y \ddot{x}_g + v \tag{2}$$

$$z = C_z x + D_z u + F_z \ddot{x}_g \tag{3}$$

where x is the state vector, $\ddot{x}_g$ is the scalar ground acceleration, u is the scalar control input, y is a vector of responses that can directly be measured $= [x_m\ \ddot{x}_{a1}\ \ddot{x}_{a2}\ \ddot{x}_{a3}\ \ddot{x}_{am}\ \ddot{x}_g]^t$, z is a vector of responses that can be regulated $= [x_1\ x_2\ x_3\ x_m\ \dot{x}_1\ \dot{x}_2\ \dot{x}_3\ \dot{x}_m\ \ddot{x}_{a1}\ \ddot{x}_{a2}\ \ddot{x}_{a3}\ \ddot{x}_m]^t$, x_i is the displacement of the ith floor relative to the ground, x_m is the displacement of the AMD relative to the third floor; $\ddot{x}_{ai}$ is the absolute acceleration of the AMD, v is a vector of measurement noises and A, B, E, C_y, D_y, C_z, D_z, F_y and F_z are coefficient matrices of appropriate dimensions obtained from experimental data as SDC/EEL.

4. TIME-DELAY MODEL

Time-delay factors arise mainly from two sources: The first stems from finite processing of information transfer due to the use of digital equipments. The second source is due to the limited nature of sensor data gathering and actuator signal processing. To account for these sources, we cast the model experimental setup into the following model:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + B_0 u(t) + A_1 x(t-d) + B_1 u(t-h) + D_0 w(t) \\ y(t) &= C_0 x(t) + L_0 u(t) + v(t) \\ \mathrm{z}(t) &= E_0 x(t), \quad x(t) = \psi(t) \quad \forall t \in [-\max(d,h), 0] \end{aligned} \tag{4}$$

where $t \in \Re$ is the time, $x \in \Re^n$ is the state, $u \in \Re^m$ is the control input, $w \in \Re^q$ is the input disturbance which belongs to $L_2[0, \infty)$, $z \in \Re^p$ and $d, h \geqslant 0$ representing the amount of delay in the state and at the input of the system, respectively. The matrices $A_0 \in \Re^{n\times n}$, $B_0 \in \Re^{n\times m}$, represent the nominal system without delay where the pair (A_0, B_0) are controllable and $A_1 \in \Re^{n\times n}$, $B_1 \in \Re^{n\times m}$, $D_0 \in \Re^{n\times q}$ are known matrices and $\psi(t) \in C[-\max(d,h), 0]$ is a continuous vector valued initial function. In the sequel, the delay factors are taken different, that is $d \neq h$. The relationships of model (1)–(3) to the time-delay model(4) is rather obvious. The control objective to derive the feedback control law

$$u(t) = \Psi[x(t)] \tag{5}$$

so that the closed-loop system (4) and (5) is stabilized and the effect of disturbance is reduced to a prespecified level. Specifically, the objective is to achieve a desirable performance in the H_∞-framework.

Definition. The time-delay system (4) is said to be *stabilizable* with a degree of stability $\psi > 0$ if there exists a feedback control $\Psi[x(t)]$ such that the resulting closed-loop system is stable and the solution of the controlled system satisfies

$$\|x(\tau_a)\| \leqslant \|x(\tau_b)\| \exp[-\psi(\tau_a - \tau_b)], \quad \forall \tau_a \geqslant \tau_b \in \Re_+ \tag{6}$$

One of the popular forms of equation (5) is the linear constant-gain state-feedback in which

$$\Psi[x(t)] = Fx(t), \quad F \in \Re^{m\times n} \tag{7}$$

Applying the control (7) to equation (4) yields the closed-loop system

$$\begin{aligned} x(t) &= A_c x(t) + A_1 x(t-d) + B_1 Fx(t-h) + D_0 w(t) \\ z(t) &= E_0 x(t), \quad A_c = A_0 + B_0 F \end{aligned} \tag{8}$$

for which the transfer function in s-domain from the disturbance $w(t)$ to the output $z(t)$ is given by

$$\begin{aligned} T_{zw}(s) &= E_0\,[(sI - A_c) - (A_1 e^{-ds} + B_1 F e^{-hs})]^{-1}\, D_0 \\ &= E_0\,[(sI - [A_0 + B_0 F]) - (A_1 e^{-ds} + B_1 F e^{-hs})]^{-1}\, D_0 \end{aligned} \tag{9}$$

Of particular interest here is the H_∞-controller. That is, the basis of designing controller (7) is to simultaneously stabilize equation (8) and to guarantee the H_∞ norm bound γ of the closed-loop transfer function T_{zw}, namely $\|T_{zw}\|_\infty \leqslant \gamma;\ \gamma > 0$.

5. H_∞-STATIC STATE FEEDBACK CONTROL

In the following, conditions are developed for stabilizing (8) and guaranteeing the desired H_∞ norm bound using Lyapunov stability theory.

Theorem 1. The closed-loop system (8) *is stabilizable with degree* $\beta > 0$ *for all* $d, h \geqslant 0$ *if one of the following equivalent conditions is satisfied*:

(1) *There exist matrices* $0 < P^t = P \in \Re^{n\times n}$, $0 < Q_1^t = Q_1 \in \Re^{n\times n}$, $0 < Q_2^t = Q_2 \in \Re^{n\times n}$ *solving the LMI*:

$$W_1 = \begin{bmatrix} P(A_c - \beta I) + (A_c - \beta I)^t P + Q_1 + Q_2 & \cdot & PA_1 & PB_1F \\ \cdots & \cdot & \cdots & \cdots \\ A_1^t P & \cdot & -R_1 & 0 \\ F^t B_1^t P & \cdot & 0 & -R_2 \end{bmatrix} < 0 \tag{10}$$

$R_1 = e^{2\beta d} Q_1, \quad R_2 = e^{2\beta h} Q_2$

(2) *There exist matrices* $0 < P^t = P \in \Re^{n\times n}$, $0 < Q_1^t = Q_1 \in \Re^{n\times n}$, $0 < Q_2^t = Q_2 \in \Re^{n\times n}$ *solving the algebraic Riccati inequality* (*ARI*):

$$P(A_0 + B_0F - \beta I) + (A_0 + B_0F - \beta I)^t P + PA_1R_1^{-1}A_1^tP + PB_1FR_2^{-1}F^tB_1^tP + Q_1 + Q_2 < 0 \tag{11}$$

Proof. (1) Define the quadratic Lyapunov function candidate $V(\cdot): \Re^n \times \Re_+ \to \Re_+$ as

$$V(x,t) = e^{-2\beta t} x^t(t) P x(t) + \int_{t-d}^{t} e^{-2\beta\vartheta} x^t(\vartheta) Q_1 x(\vartheta)\, d\vartheta + \int_{t-h}^{t} e^{-2\beta\vartheta} x^t(\vartheta) Q_2 x(\vartheta)\, d\vartheta \tag{12}$$

where $Q_1, Q_2 > 0$. Observe that $V(x,t) > 0,\ x \neq 0$; $V(x,t) = 0,\ x = 0$ and more importantly that

$$\begin{aligned} \lambda_m(P)\ \|x\|^2 e^{-2\beta t} \ &\leqslant\ V(x,t)\ \leqslant\ [\lambda_M(P) + \Psi_{ss}]\ \|x\|^2 e^{-2\beta t} \\ \Psi_{ss} &= e^{2\beta d}\lambda_M(Q_1) + e^{2\beta h}\lambda_M(Q_2) \end{aligned} \tag{13}$$

Evaluating the derivative of equation (12) along the solutions of equation (8) results in

$$\begin{aligned} \dot V(x,t) &= e^{-2\beta t}(x^t(t)[P(A_c - \beta I) + (A_c - \beta I)^t P + Q_1 + Q_2]x(t)) \\ &\quad + e^{-2\beta t}(x^t(t)PA_1x(t-d) + x^t(t)PB_1Fx(t-h)) \\ &\quad + e^{-2\beta t}(x^t(t-d)A_1^tPx(t) + x^t(t-h)\,F^tB_1^tPx(t)) \\ &\quad - e^{-2\beta t}(e^{2\beta d}x^t(t-d)Q_1x(t-d) + e^{2\beta h}x^t(t-h)Q_2x(t-h)) \\ &= e^{-2\beta t}\chi^t(t) W_1 \chi(t) \\ \chi^t(t) &= [x^t(t) \quad x^t(t-d) \quad x^t(t-h)] \end{aligned} \tag{14}$$

When condition (10) is satisfied, then $\dot{V}(x,t)<0$. From equation (11), it can be shown that the closed-loop state trajectories satisfy the state-norm inequality

$$\|x(\xi)\| \leqslant \sqrt{\frac{\lambda_{\mathrm{M}}(P)+\Psi_{ss}}{\lambda_{\mathrm{m}}(P)}}\;\|x(\pi)\|\mathrm{e}^{-\beta(\xi-\pi)}, \quad \forall\xi\geqslant\pi \tag{15}$$

In view of equation (13), we conclude that the closed-loop system (5) is exponentially stable with degree $\beta>0$ for all $d, h\geqslant 0$ as desired.

(2) Given equation (10) and using the Schur complement, we can easily get equation (11).

Remark 1. Theorem 1 is new for time-delay systems and provides a delay-dependent stability criteria since condition (10) includes the delay factors d and h. However, it presumes the availability of gain matrix F.

Now, we establish the conditions under which the controller (7) stabilizes (8) and guarantees the H_∞ norm γ of the closed-loop transfer function T_{zw}, namely $\|T_{zw}\|_\infty\leqslant\gamma$; $\gamma>0$.

Theorem 2. The closed-loop system (8) *is stabilizable with degree $\beta>0$ and $\|T_{zw}\|_\infty\leqslant\gamma$; $\gamma>0$. for all d, $h\geqslant 0$ if there exist matrices $0<P^{\mathrm{t}}=P\in\Re^{n\times n}$, $0<Q_1^{\mathrm{t}}=Q_1\in\Re^{n\times n}$ and $0<Q_2^{\mathrm{t}}=Q_2\in\Re^{n\times n}$ solving the ARI*:

$$\begin{aligned}&P(A_0+B_0F-\beta I)+(A_0+B_0F-\beta I)^{\mathrm{t}}P+Q_1+Q_2\\&\quad+PA_1R_1^{-1}A_1^{\mathrm{t}}P+PB_1FR_2^{-1}F^{\mathrm{t}}B_1^{\mathrm{t}}P+E_0^{\mathrm{t}}E_0+\gamma^{-2}PD_0D_0^{\mathrm{t}}P<0\end{aligned} \tag{16}$$

Proof. See Reference 17.

Remark 2. The usefulness of Theorem 2 lies in the fact that it provides condition (16) as the sufficient measure to select a constant matrix F as the gain of an H_∞-controller. However, it has a general form and is not applicable to typical engineering systems. Theorem 3 below is therefore a further development of Theorem 2 and can be applied to the benchmark problem. In effect, the latter theorem provides a delay-dependent LMI condition for a memoryless H_∞-controller which guarantees the norm bound γ of the transfer function T_{zw}.

Theorem 3. There exists a memoryless state-feedback controller such that the closed-loop system (5) *is stabilizable with degree $\beta>0$ and $\|T_{zw}\|_\infty\leqslant\gamma$; $\gamma>0$ for all d, $h\geqslant 0$ if and only if there exist matrices $0<Y^{\mathrm{t}}=Y\in\Re^{n\times n}$, $0<Q_{\mathrm{t}}^{\mathrm{t}}=Q_{\mathrm{t}}\in\Re^{n\times n}$, $0<Q_{\mathrm{s}}=Q_{\mathrm{s}}^{\mathrm{t}}\in\Re^{n\times n}$, $S\in\Re^{m\times n}$; solving the LMI*:

$$\begin{bmatrix} \begin{matrix}A_{\mathrm{s}}Y+YA_{\mathrm{s}}^{\mathrm{t}}+B_0S+S^{\mathrm{t}}B_0^{\mathrm{t}}\\ +Q_{\mathrm{t}}+Q_{\mathrm{s}}\end{matrix} & \cdot & YE^{\mathrm{t}} & B_1S & D & A_1Y\\ \cdots & \cdots & \cdot & \cdots & \cdots & \cdots\\ EY & \cdot & -I & 0 & 0 & 0\\ S^{\mathrm{t}}B_1^{\mathrm{t}} & \cdot & 0 & -\mathrm{e}^{2\beta h}Q_{\mathrm{s}} & 0 & 0\\ D^{\mathrm{t}} & \cdot & 0 & 0 & -\gamma^2 I & 0\\ YA_1^{\mathrm{t}} & \cdot & 0 & 0 & 0 & -\mathrm{e}^{2\beta d}Q_{\mathrm{t}} \end{bmatrix}<0 \tag{17}$$

where $A_{\mathrm{s}}=A_0-\beta I$. Moreover, the gain of the memoryless state-feedback controller is given by

$$F=SY^{-1} \tag{18}$$

Proof. ($\Rightarrow$) By Theorem 2,[18] there exists a state-feedback controller with constant gain F such that the closed-loop system (8) is stabilizable with degree $\beta>0$ and $\|T_{zw}\|_\infty\leqslant\gamma$; $\gamma>0$ for all $d, h\geqslant 0$. Now to ensure

the convexity of equation (16) in P and F, we let $Y = P^{-1}, S = FY, Y = P^{-1}, S = FY, Q_t = P^{-1}Q_1P^{-1}$, $Q_s = P^{-1}Q_2P^{-1}$. Premultiplying equation (16) by P^{-1} and postmultiplying the result by P^{-1}, we get

$$\begin{aligned} & A_sY + YA_s^t + Q_t + Q_s + B_0S + S^tB_0^t \\ & A_1Y\mathrm{e}^{-2\beta d}Q_t^{-1}YA_1^t + B_1S\mathrm{e}^{-2\beta h}Q_s^{-1}S^tB_1^t + YE^tEY + \gamma^{-2}DD^t < 0 \end{aligned} \tag{19}$$

which, in the light of Schur complement, can be conveniently arranged to yield the block form equation (17) as desired.

($\Leftarrow$) This follows easily by reversing the foregoing steps and using the memoryless state-feedback controller $F = SY^{-1}$.

To obtain such a controller, one has to solve the following minimization problem:

$$\begin{aligned} & \underset{Y,S,Q_s,Q_t}{\mathrm{Min}} \ \gamma \\ & \text{subject to } Y > 0, Q_s > 0, Q_t > 0 \text{ and equation (17)} \end{aligned} \tag{20}$$

6. H_∞-TWO-TERM FEEDBACK CONTROL

As a departure from the memoryless state feedback, we now propose the controller

$$u(t) = Fx(t) + Kx(t-d) \tag{21}$$

which combines the effect of the instantaneous as well as the delay states. In some sense, it can be called a PD controller since it provides two degrees of freedom: one by the proportional (P) term (Fx) and the other through the delay (D) term $Kx(t-d)$. The controller (21) plus system (4) gives the closed-loop system:

$$\begin{aligned} x(t) &= A_cx(t) + A_dx(t-d) + B_1Fx(t-h) + B_1Kx(t-d-h) + D_0w(t) \\ z(t) &= E_0x(t) \\ A_c &= A_0 + B_0F, \quad A_d = A_1 + B_0K \end{aligned} \tag{22}$$

The transfer function from the disturbance $w(t)$ to the output $z(t)$ is given by

$$T_{zw}(s) = E_0[(sI - A_c) - (A_d\mathrm{e}^{-ds} + B_1F\mathrm{e}^{-hs} + B_1K\mathrm{e}^{-(d+h)s})]^{-1}D_0 \tag{23}$$

To study the stability behavior in this case, we define a quadratic Lyapunov function candidate as

$$\begin{aligned} V(x,t) = {} & \mathrm{e}^{-2\beta t}x^t(t)\,Px(t) + \int_{t-d}^{t}\mathrm{e}^{-2\beta\vartheta}x^t(\vartheta)\,Q_1x(\vartheta)\,d\vartheta + \int_{t-h}^{t}\mathrm{e}^{-2\beta\vartheta}x^t(\vartheta)\,Q_2x(\vartheta)\,d\vartheta \\ & + \int_{t-d-h}^{t}\mathrm{e}^{-2\beta\vartheta}x^t(\vartheta)\,Q_3x(\vartheta) \end{aligned} \tag{24}$$

and observe that

$$\begin{aligned} & \lambda_m(P)\,\|x\|^2\mathrm{e}^{-2\beta t} \leqslant V(x,t) \leqslant [\lambda_M(P) + \Psi_{dd}]\,\|x\|^2\mathrm{e}^{-2\beta t} \\ & \Psi_{dd} = \mathrm{e}^{2\beta d}\lambda_M(Q_1) + \mathrm{e}^{2\beta h}\lambda_M(Q_2)\ \mathrm{e}^{2\beta(d+h)}\ \lambda_M(Q_3) \end{aligned} \tag{25}$$

Theorem 4. The closed-loop system (22) *is stabilizable with degree $\beta > 0$ for all d, $h \geqslant 0$ if one of the following equivalent conditions is satisfied*:

(1) *There exist matrices* $0 < P^t = P \in \Re^{n\times n}$, $0 < Q_1^t = Q_1 \in \Re^{n\times n}$, $0 < Q_2^t = Q_2 \in \Re^{n\times n}$ *and* $0 < Q_3^t = Q_3 \in \Re^{n\times n}$ *solving the LMI*:

$$W_2 = \begin{bmatrix} P(A_c - \beta I) + (A_c - \beta I)^t P + Q_1 + Q_2 + Q_3 & \cdot & PA_d & PB_1 F & PB_1 K \\ \cdots & \cdot & \cdots & \cdots & \cdots \\ A_d^t P & \cdot & -R_1 & 0 & 0 \\ F^t B_1^t P & \cdot & 0 & -R_2 & 0 \\ K^t B_1^t P & \cdot & 0 & 0 & -R_3 \end{bmatrix} < 0 \tag{26}$$

$$R_1 = e^{2\beta d} Q_1, \qquad R_2 = e^{2\beta h} Q_2, \qquad R_3 = e^{2\beta (d+h)} Q_3$$

(2) *There exist matrices* $0 < P^t = P \in \Re^{n\times n}$, $0 < Q_1^t = Q_1 \in \Re^{n\times n}$, $0 < Q_2^t = Q_2 \in \Re^{n\times n}$ *and* $0 < Q_3^t = Q_3 \in \Re^{n\times n}$ *solving the ARI*:

$$\begin{aligned} &P(A_0 + B_0 F - \beta I) + (A_0 + B_0 F - \beta I)^t P + P(A_1 + B_0 F) R_1^{-1} (A_1 + B_0 F)^t P \\ &\quad + PB_1 F R_2^{-1} F^t B_1^t P + PB_1 K R_3^{-1} K^t B_1^t P + Q_1 + Q_2 + Q_3 < 0 \end{aligned} \tag{27}$$

Proof. First, observe that $V(x, t) > 0$, $x \neq 0; V(x, t) = 0$, $x = 0$. The Lyapunov derivative $\dot{V}(x, t)$ evaluated along the solutions of system (22) is given by

$$\begin{aligned} \dot{V}(x, t) &= e^{-2\beta t}(x^t(t) \lfloor P(A_c - \beta I) + (A_c - \beta I)^t P + Q_1 + Q_2 + Q_3 \rfloor x(t)) \\ &\quad + e^{-2\beta t}(x^t(t) P(A_1 + B_0 F) x(t-d) + x^t(t) PB_1 F x(t-h) + x^t(t) PB_1 K x(t-d-h)) \\ &\quad + e^{-2\beta t}(x^t(t-d)(A_1 + B_0 F)^t P x(t) + x^t(t-h) F^t B_1^t P x(t) + x^t(t-d-h) K^t B_1^t P x(t)) \\ &\quad - e^{-2\beta t}(e^{2\beta d} x^t(t-d) Q_1 x(t-d) + e^{2\beta h} Q_2 x(t-h) + e^{2\beta(d+h)} x^t(t-d-h) Q_3 x(t-d-h)) \\ &= e^{-2\beta t} \chi^t(t) W_2 \chi(t) \end{aligned} \tag{28}$$

$$\chi^t(t) = [x^t(t) \quad x^t(t-d) \quad x^t(t-h) \quad x^t(t-d-h)]$$

When condition (26) is satisfied, then $\dot{V}(x, t) < 0$. From Equation (22), it can be shown that the closed-loop state trajectories satisfy the state-norm inequality

$$\|x(\xi)\| \leqslant \sqrt{\frac{\lambda_M(P) + \Psi_{dd}}{\lambda_m(P)}}\, \|x(\pi)\| e^{-\beta(\xi - \pi)}, \quad \forall \xi \geqslant \pi \tag{29}$$

In view of Equation (25), we conclude that the closed-loop system (22) is exponentially stable with degree $\beta > 0$ for all d, $h \geqslant 0$ as desired.

(2) Again, given Equation (27) and using the Schur complement, we can easily get Equation (28).

Remark 3: In a similar way, Theorem 4 is new for time-delay systems using a two-term feedback control and provides a delay-dependent stability criteria since condition (26) includes the delay factors d and h. Here, it needs the gain matrices F and K. However, it has a general form and is not applicable to typical engineering systems. Theorem 5 below is therefore a further development of Theorem 4 for two-term feedback control design and can be applied to the benchmark problem.

Theorem 5: *There exists a memoryless state-feedback controller such that the closed-loop system* (29) *is stabilizable with degree $\beta > 0$ and $\|T_{zw}\|_\infty \leqslant \gamma$; $\gamma > 0$ for all d, $h \geqslant 0$ if and only if* $0 < Y^t = Y \in \Re^{n\times n}$,

$0 < Q_t^t = Q_t \in \Re^{n\times n}$, $0 < Q_s = Q_s^t \in \Re^{n\times n}$, $0 < Q_r = Q_r^t \in \Re^{n\times n}$, $S,\ V \in \Re^{m\times n}$ and scalars $\sigma, \kappa > 0$ solving the LMI:

$$\begin{bmatrix}
\begin{array}{l} A_sY + YA_s^t + B_0S + S^tB_0^t \\ + Q_t + Q_s + Q_r \end{array} & \cdot & A_1Y & B_0S & B_1S & B_1V & YE^t & D \\
\cdot & \cdot & \cdots & \cdots & \cdots & \cdots & \cdots & + Q_r \\
YA_1^t & \cdot & -\sigma^{-1}e^{2\beta d}Q_t & 0 & 0 & 0 & 0 & 0 \\
S^tB_0^t & \cdot & 0 & -\kappa^{-1}e^{2\beta d}Q_t & 0 & 0 & 0 & 0 \\
S^tB_1^t & \cdot & 0 & 0 & -e^{2\beta h}Q_s & 0 & 0 & 0 \\
V^tB_1^t & \cdot & 0 & 0 & 0 & -e^{2\beta(d+h)}Q_r & 0 & 0 \\
EY & \cdot & 0 & 0 & 0 & 0 & -I & 0 \\
D^t & \cdot & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I
\end{bmatrix} < 0 \quad (30)$$

where $A_s = A_0 - \beta I$. Moreover, the gain of the delayed state-feedback controller is given by

$$[F \quad K] = [SY^{-1} \quad VY^{-1}] \tag{31}$$

Proof: $(\Rightarrow)$ By Theorem 2,[18] there exists a delayed state-feedback controller with constant gain $[F \quad K]$ such that the closed-loop system (22) is stabilizable with degree $\beta > 0$ for all $d, h \geqslant 0$ and $\|T_{zw}\|_\infty \leqslant \gamma;\ \gamma > 0$.

Now, letting $Y = P^{-1}$, $S = FY$, $V = KY$, $Q_t = P^{-1}Q_1P^{-1}$, $Q_s = P^{-1}Q_2P^{-1}$, and $Q_r = P^{-1}Q_3P^{-1}$, premultiplying (27) by P^{-1} and postmultiplying the result by P^{-1}, we get

$$\begin{aligned} &A_sY + YA_s^t + Q_t + Q_s + Q_r + B_0S + S^tB_0^t + (A_1 + B_0F)R_1^{-1}(A_1 + B_0F)^t \\ &+ B_1Se^{-2\beta h}Q_s^{-1}S^tB_1^t + B_1Ve^{-2\beta(d+h)}Q_r^{-1}V^tB_1^t + YE_0^tE_0Y + \gamma^{-2}D_0D_0^t < 0 \end{aligned} \tag{32}$$

Using standard matrix lemma in the term $(A_1 + B_0F)R_1^{-1}(A_1 + B_0F)^t$ and manipulating, it becomes.

$$\begin{aligned} (A_1 + B_0F)R_1^{-1}(A_1 + B_0F)^t &= A_1R_1^{-1}A_1^t + B_0FR_1^{-1}F^tB_0^t + B_0FR_1^{-1}A_1^t + A_1R_1^{-1}F^tB_0^t \\ &\leqslant (1 + \alpha)A_1R_1^{-1}A_1^t + (1 + \alpha^{-1})B_0FR_1^{-1}F^tB_0^t \\ &\leqslant \sigma A_1Ye^{2\beta d}Q_t^{-1}YA_1^t + \kappa B_0SQ_t^{-1}S^tB_0^t \quad \forall \sigma, \kappa > 0 \end{aligned} \tag{33}$$

so that Equation (27) can be expressed as

$$\begin{aligned} &A_sY + YA_s^t + Q_t + Q_s + Q_r + B_0S + S^tB_0^t + \sigma A_1Te^{-2\beta d}Q_t^{-1}YA_1^t + \gamma^{-2}DD^t \\ &\kappa B_0Se^{-2\beta d}Q_t^{-1}S^tB_0^t + B_1Se^{-2\beta h}Q_s^{-1}S^tB_1^t + B_1Ve^{-2\beta(d+h)}\ Q_r^{-1}V^tB_1^t + YE^tEY < 0 \end{aligned} \tag{34}$$

which, in the light of Schur complement, can be conveniently arranged to yield the block form Equation (30) as desired.

$(\Leftarrow)$ This follows easily be reversing the foregoing steps and using the delayed state-feedback controller with constant gain $[F\,K] = [SY^{-1}\ VY^{-1}]$.

Remark 4: Theorem 5 provides a delay-dependent LMI condition for two-term H_∞-controller which guarantees the norm bound γ of the transfer function T_{zw}.

To determine the gains of such a controller, one has to solve the following minimization problem:

$$\begin{aligned} &\underset{Y,S,V,Q_s,Q_t,Q_r,\sigma,\kappa}{\mathrm{Min}} && \gamma \\ &\text{subject to} && Y > 0, Q_s > 0, Q_t > 0, Q_r > 0, \sigma > 0, \kappa > 0 \text{ and Equation (30)} \end{aligned} \tag{35}$$

Remark 5: It is significant to note that the minimization problem addressed in Remark 2 has the form of a generalized eigenvalue problem which is known to be solvable numerically very efficiently using interior-point methods.[17] The software LMI-Control Toolbox[19] provides an efficient tool for computer implementation. Although the same goes to the minimization problem (35) addressed in Remark 4, however a special note is called for. When using the LMI-Control Toolbox, a prerequisite is that the block 11 (upper left block) should contain a symmetric term of the variables involved. In our case the term corresponding to the gain V is absent. Direct implementation of the LMI-Control Toolbox would then yield $V = 0$. To get around this difficulty, we have added the term $T_1(\varepsilon)\, V T_2(\varepsilon) + T_1^t(\varepsilon)\, V T_2^t(\varepsilon)$ and consider $\varepsilon \to 0$ where the matrices $T_1(\varepsilon)$, $T_2(\varepsilon)$ are matrices with appropriate dimensions. An alternative way would be to repeatedly assign V and then solve Equation (35) for Y and S. The next step is to choose the one yielding the minimum of the resulting solutions. This is obviously a tedious process and will not be followed here.

7. H_∞-STATIC OUTPUT FEEDBACK CONTROL

Now, we consider system (1) when a limited number of states are accessible for measurement. In this regard, we recall the output measurement

$$y(t) = C_o x(t) \tag{36}$$

where $y \in \Re^s$ is the measured output and $C_o \in \Re^{s\times n}$ is a constant matrix such that the pair (A_o, C_o) is observable. Our purpose is to develop an output feedback control for system (4) and (36) of the form $u(t) = \Psi[y(t)]$. In this section, the control law is given by

$$u(t) = Gy(t) = GC_o x(t) \tag{37}$$

where G is a static gain matrix to be determined. It should be emphasized that controller (37) can be considered as a replica of the state-feedback controller (8) and as such no generality is claimed at this point. In Reference 20, it was clearly stated that results pertaining to output-feedback synthesis for dynamical systems are only few and restrictive. Here we propose the gain

$$GC_o = \omega B_o^t P + \Omega, \quad \omega > 0 \tag{38}$$

The closed-loop output feedback system of (4), (36) and (37) is given by

$$\begin{aligned} \dot{x}(t) &= (A_o + B_o G C_o)x(t) + A_1 x(t-d) + B_1 G C_o x(t-h) + D_o w(t) \\ z(t) &= E_o x(t) \end{aligned} \tag{39}$$

the following theorem summarizes the desired result:

Theorem 6: *There exists a memoryless feedback controller such that the closed-loop system* (39) *is stabilizable with degree $\beta > 0$ and $\|T_{zw}\|_\infty \leqslant \gamma;\ \gamma > 0$ for all $d,\ h \geqslant 0$ if and only if $0 < Y^t = Y \in \Re^{n\times n}$, $0 < Q_t^t = Q_t \in \Re^{n\times n}$, $0 < Q_s = Q_s^t \in \Re^{n\times n}$, $\Gamma \in \Re^{m\times n}$ and scalar $\omega > 0$ solving the LMI*:

$$\begin{bmatrix} \begin{matrix} A_s Y + Y A_s^t + B_o\Gamma + \Gamma^t B_0^t \\ + Q_t + Q_s + 2\omega B_o B_o^t \end{matrix} & \cdot & A_1 Y & \omega B_1 B_o^t & B_1\Gamma & YE^t & D \\ \cdots & \cdot & \cdots & \cdots & \cdots & \cdots & \cdots \\ YA_1^t & \cdot & -Q_{ot} & 0 & 0 & 0 & 0 \\ \omega B_o B_1^t & \cdot & 0 & -Q_{os} & 0 & 0 & 0 \\ \Gamma^t B_1^t & \cdot & 0 & 0 & -Q_{os} & 0 & 0 \\ EY & \cdot & 0 & 0 & 0 & -I & 0 \\ D^t & \cdot & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{40}$$

$$Q_{ot} = e^{2\beta d} Q_t,\ Q_{os} = e^{2\beta h} Q_s$$

where $A_s = A_o - \beta I$. Moreover, the constant gain of the feedback controller is given by

$$F = [\omega B_o^t Y^{-1} + \Gamma Y^{-1}] \tag{41}$$

Proof: ($\Rightarrow$) By Theorem 2 Reference [18], there exists a memoryless feedback controller with constant gain $F = GC_o = \omega B_o^t P + \Omega$ such that the closed-loop system (39) is stabilizable with degree $\beta > 0$ for all d, $h \geqq 0$ and $\|T_{zw}\|_\infty \leqslant \gamma$; $\gamma > 0$. The stabilizing controller satisfies the inequality:

$$P(A_o + B_o GC_o - \beta I) + (A_o + B_o GC_o - \beta I)^t P + Q_1 + Q_2 + PA_1 R_1^{-1} A_1^t P \\ + PB_1 GC_o R_2^{-1} C_o^t G^t B_1^t P + E_o^t E_o + \gamma^{-2} PD_o D_o^t P < 0 \tag{42}$$

Now, letting $Y = P^{-1}$, $\Omega = \Gamma P$, $Q_t = P^{-1} Q_t P^{-1}$, $Q_s = P^{-1} Q_2 P^{-1}$, premultiplying (42) by P^{-1} and postmultiplying the result by P^{-1}. we get

$$A_s Y + YA_s^t + Q_t + Q_s + B_o \Gamma + \Gamma^t B_o^t + 2\omega B_o B_o^t + A_1 Y e^{-2\beta d} Q_t^{-1} Y A_1^t \\ + YE^t EY + \gamma^{-2} DD^t + \omega^2 B_1 B_o^t e^{-2\beta h} Q_s^{-1} B_o B_1^t + B_1 \Gamma e^{-2\beta h} Q_s^{-1} \Gamma^t B_1^t < 0 \tag{43}$$

which, in the light of Schur complement, can be conveniently arranged to yield the block form (40) as desired.

($\Leftarrow$) This follows easily by reversing the foregoing steps and using the memoryless feedback controller with constant gain $\omega B_o^t Y^{-1} + \Gamma Y^{-1}$.

Remark 6: Theorem 6, like Theorem 3, is a further development of Theorem 2 for the output feedback control design of time-delay systems which is applicable to the benchmark problem.

To determine the gain factors ω, Γ, Y one has to solve the following problem:

$$\begin{array}{ll} \underset{Y,\Gamma,V,Q_s,Q_t,\omega}{\text{Min}} & \gamma \\ \text{subject to} & Y > 0, Q_s > 0, Q_t > 0, \omega > 0 \text{ and Eq. (40)} \end{array} \tag{44}$$

8. A COMPUTATIONAL PROCEDURE

In order to apply the developed feedback control design methods to the reduced-order (10-states) model of the benchmark problem, the following computational procedure has been adopted:

Step 1: READ In the system matrices $A_o = A_r$, $B_o = B_r$, $C_o = C_{yr}$, $D_o = E_r$
Step 2: SELECT the system matrices $A_1 = 10^{-3} \times I_{10}$, $B_1 = 10^{-3} \times e_{10}$, $E_o = 10^{-1} \times I_{10}$
Step 3: SET the time-delay factors at $d = 0{\cdot}0004$ and $h = 0{\cdot}0008$
Step 4: SELECT initial values for β, Q_s and Q_t
Step 5: ENTER all the values of Steps 1–4 into an M. file.

Step 6: CALL the LMI-Control Toolbox and solve Problems (20), (35) and (44), sequentially, to get the $\gamma_{\min}$ and the gain matrices of equations (18), (31) and (41), respectively. If a feasible solution is attained, go to Step 7. Otherwise, Update β, Q_s, and Q_t and go back to Step 5.

Step 7: REPLACE the gain matrix [K] employed in the sample control program simp-des by the matrix [F] obtained in Step 6.

Step 8: LOAD the M.file eval-mod, representing the evaluation model (28-states), and redefine the system matrices

$$A_0 = A;\ B_0 = [E, B]; D_0 = [[F_z, D_z]; [10]]$$
$$A_1 = 10^{-3} \times I_{28}; B_1 = [E, 10^{-3} \times e_{10}];$$
$$E_0 = 10^{-1} \times \begin{bmatrix} I_{12} & 0 \\ 0 & 0 \end{bmatrix}, \qquad C_0 = \begin{bmatrix} C_{yr} + D_{yr} K & 0 \\ 0 & 0 \end{bmatrix}, \qquad F_{yr1} = \begin{bmatrix} F_{yr} \\ \cdots \\ 0 \end{bmatrix}$$

Step 9: SIMULATE the closed-loop system using one of the input signals provided: Kanai–Tajimi filter, El-Centro and Hachinohe earthquakes. The input parameter T_f is set to 300 sec while parameters ω_g and z_g are varied in the ranges $20 \leqslant \omega_g \leqslant 120$ and $0{\cdot}30 \leqslant \zeta \leqslant 0{\cdot}75$ for the Kanai–Tajimi filter. For El-Centro and Hachinohe earthquakes, we use $T_f = 10$ and 7·2, respectively.

Step 10: USE the output vector zout to derive the evaluation indices and PLOT the vibration responses.

Remark 7: The choice of the delay parameters d and h in Step 3 has been governed by two main factors: first is the small values of the delay parameters are closer to reality and account for the finite time of information transfer and data processing. Secondly, the guaranteed convergence oof LMI-based solutions using all feedback techniques.

9. APPLICATION AND RESULTS

A SIMULINK model was originally provided with the benchmark problem[4] to simulate the response of the structure as described by equations (1)–(3). In order to simulate the response of the structure taking into account the time-delay model as represented by equation (4), the SIMULINK model has been modified by properly incorporating the block-diagram of Figure 3, corresponding to the time-delay, into the block SUBSYSTEM of Figure 2.

A study of the variation of the norm of the feedback gain matrix $\|F\|$, as a possible measure of the control effort, and $\gamma_{\min}$, the least upper bound of the infinite-norm of the closed-loop transfer function $\|T_{zw}\|_\infty$, with β and the weighting matrices Q_s, Q_t is carried out and the obtained results are plotted in Figures 4–10 for the three described feedback models. For the purpose of systematic implementation, we have used $Q_s = a_s I$,

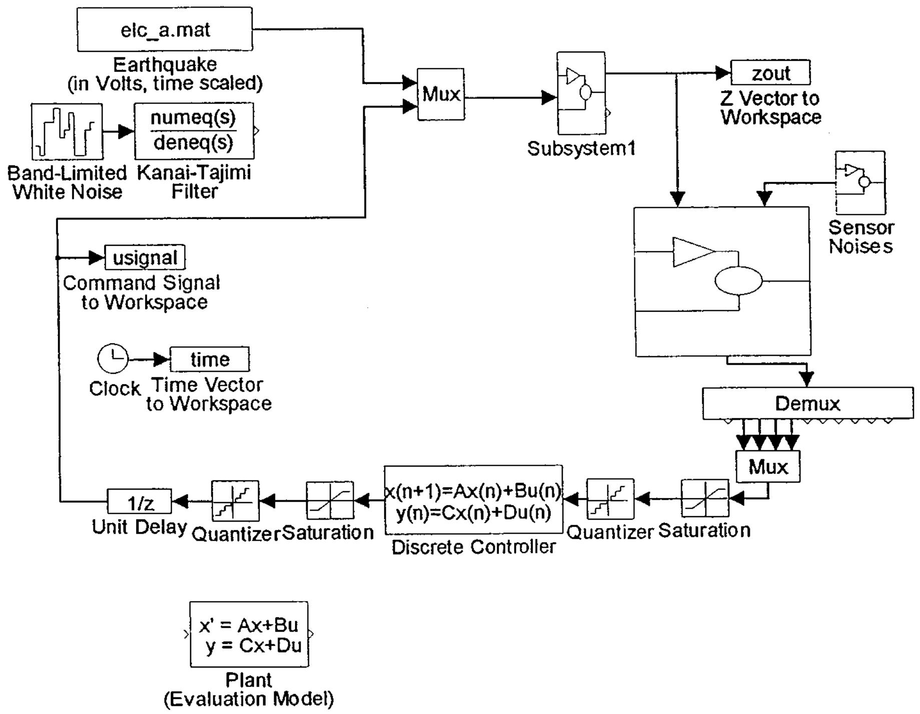

Figure 2. SIMULINK model block diagram for the Benchmark Problem

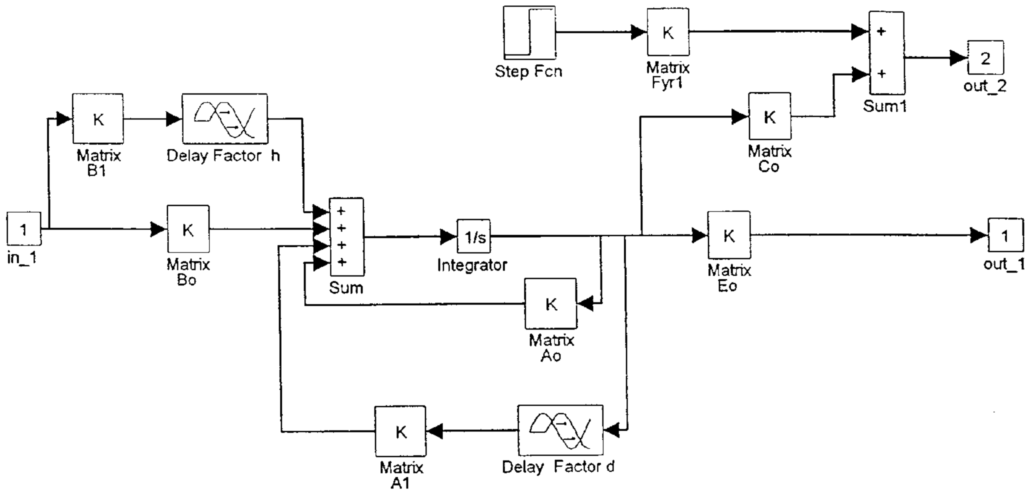

Figure 3. Block diagram for the time delayed model

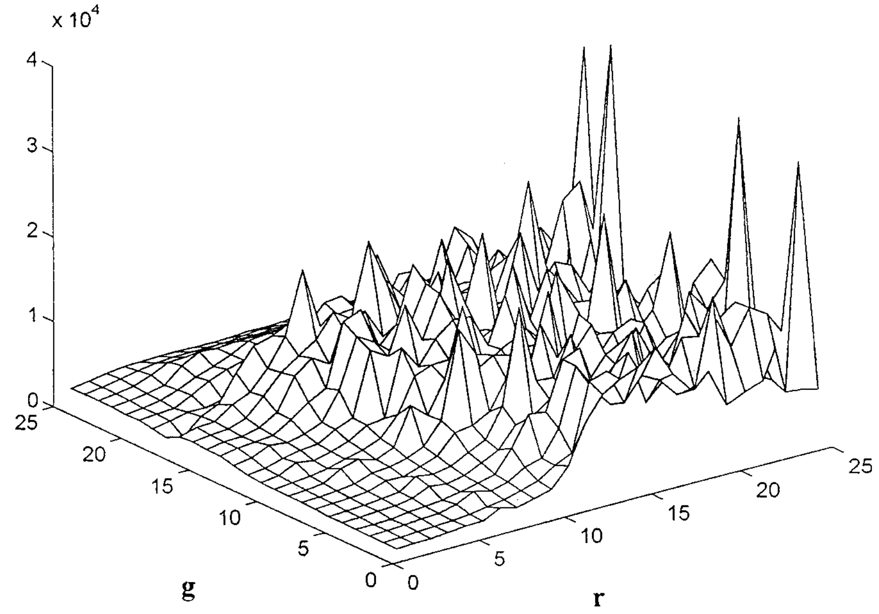

Figure 4. Variation of $\|F\|$ with $Q_s = a_s \times I$; $Q_t = a_t \times I$ and β values for the state feedback, where:

$$a_s = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ a_t = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ \beta = 5 + 0{\cdot}625 \times g$$

$Q_t = a_t I$ and considered different values of the scalars a_s, a_t. However, any other form of positive-definite matrices could also be equally used.

The value of $\gamma_{\min}$ is stable for values of β, a_s and a_t above 10, 25 and 15, respectively, whereas the values of $\|F\|$ appeared to be more affected by the change in the values of a_s and a_t. However, all values in

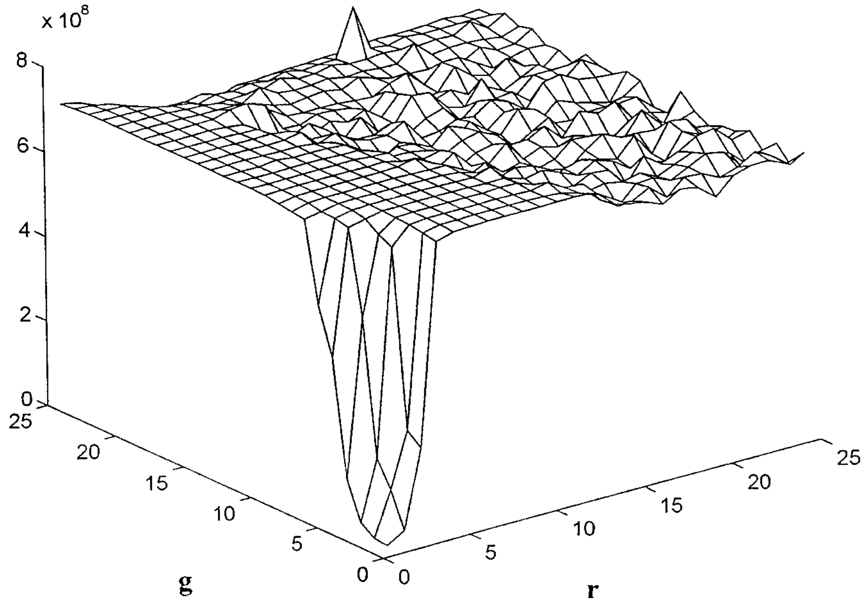

Figure 5. Variation of γ minimum with $Q_s = a_s \times I$; $Q_t = a_t \times I$ and β values for the state-delayed feedback, where:

$$a_s = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ a_t = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ \beta = 5 + 0{\cdot}625 \times g$$

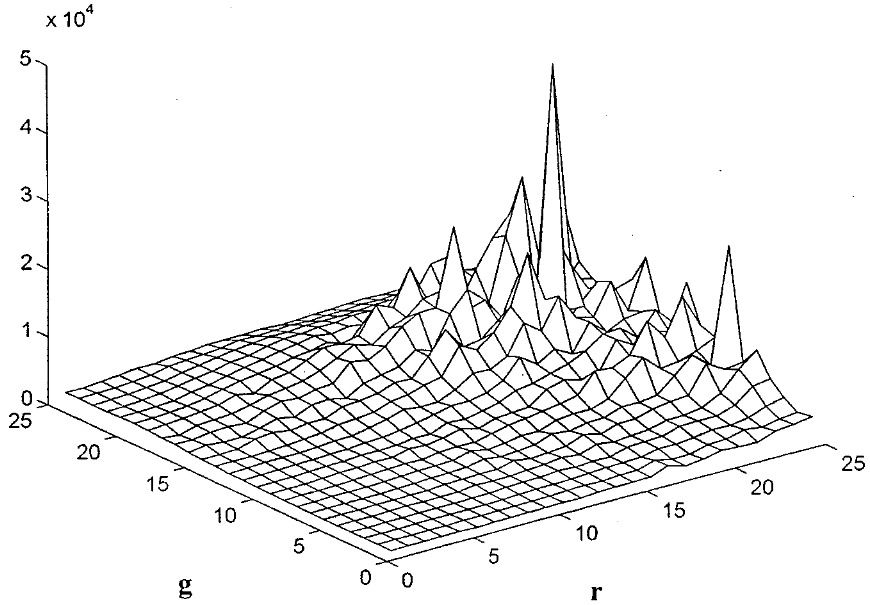

Figure 6. Variation of $\|F\|$ with $Q_s = a_s \times I$; $Q_t = a_t \times I$ and β values for the state-delayed feedback, where:

$$a_s = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ a_t = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ \beta = 5 + 0{\cdot}625 \times g$$

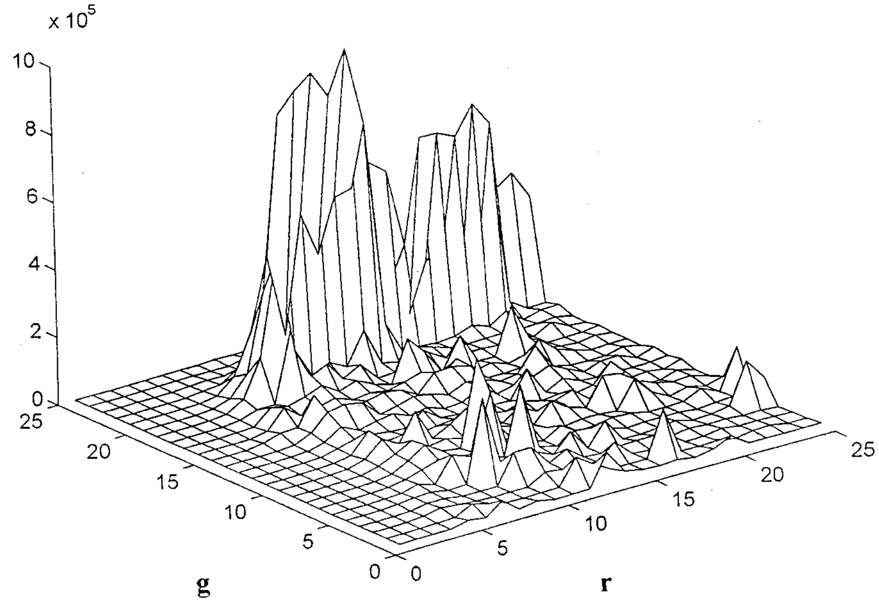

Figure 7. Variation of $\|K\|$ with $Q_s = a_s \times I$; $Q_t = a_t \times I$ and β values for the state-delayed feedback, where:

$$a_s = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ a_t = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ \beta = 5 + 0{\cdot}625 \times g$$

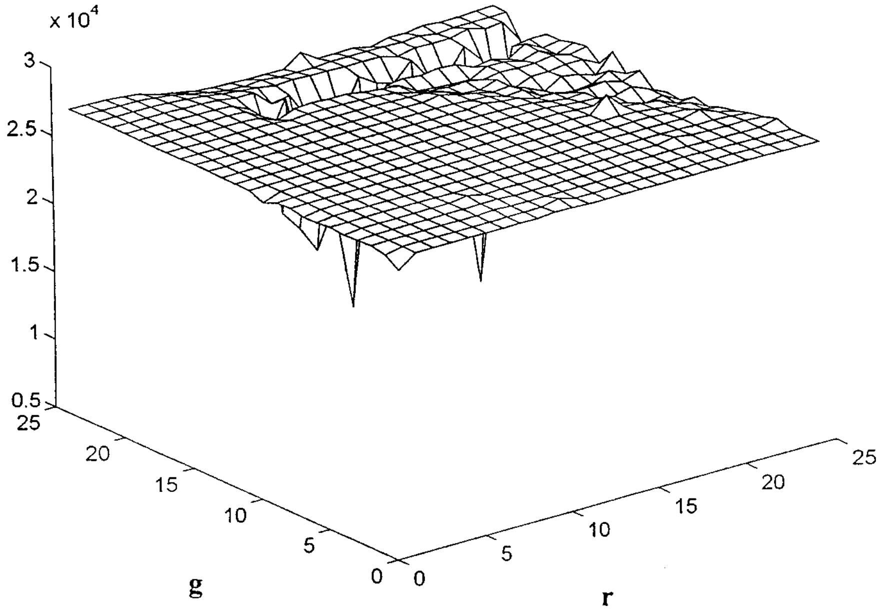

Figure 8. Variation of γ minimum with $Q_s = a_s \times I$; $Q_t = a_t \times I$ and β values for the state-delayed feedback, where:

$$a_s = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ a_t = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ \beta = 5 + 0{\cdot}625 \times g$$

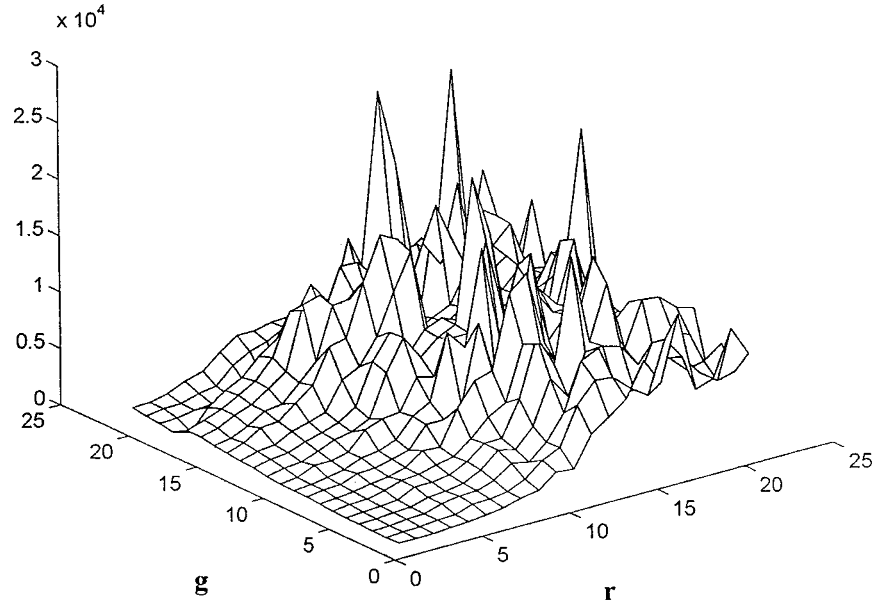

Figure 9. Variation of $\|F\|$ with $Q_s = a_s \times I$; $Q_t = a_t \times I$ and β values for the output feedback, where:

$$a_s = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ a_t = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ \beta = 5 + 0{\cdot}625 \times g$$

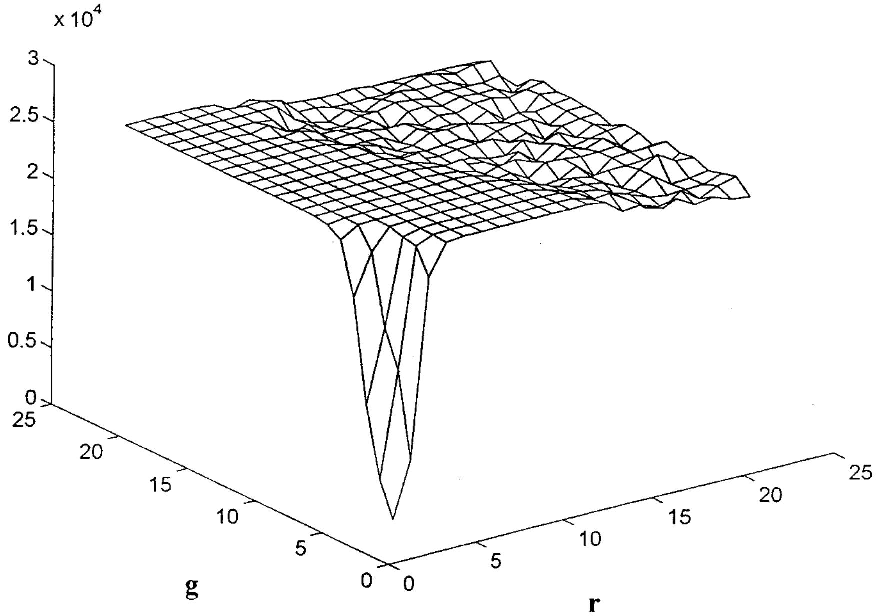

Figure 10. Variation of γ minimum with $Q_s = a_s \times I$; $Q_t = a_t \times I$ and β values for the output feedback, where:

$$a_s = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ a_t = 2{\cdot}5 \times (\sqrt[5]{10})^r;\ \beta = 5 + 0{\cdot}625 \times g$$

Figures 4–10 have yielded a feasible solution for the LMI problem definitions discussed above. For the purpose of creating a uniform mesh grid for values of β, a_s and a_t (using MATLAB), we have chosen $a_t = a_s = 2{\cdot}5 \times (\sqrt[5]{10})^r$, $\beta = 5 + 0{\cdot}625 \times g$ and changing r and g to cover the range $a_s = a_t \in [2{\cdot}5 \rightarrow 2{\cdot}5 \times 10^5]$, $\beta \in [5 \rightarrow 15]$. This choice of parameters r and g has been for graphical purposes. The adopted ranges of values β, a_s and a_t are only representative of typical feasible solutions. It should be noted that the quantity $\|F\|$ as a function of β, a_s and a_t has many extremum points from which we select a proper candidate. On the other hand, the quantity $\gamma_{\min}$ as a function of β, a_s and a_t has a distinct minimum value. For the purpose of completeness, a summary of the values of $\|F\|$ and $\gamma_{\min}$ as a function of the parameters β, a_s and a_t is shown in Tables I to III for the state-feedback, state-delayed feedback and static output feedback,

Table I. $\|F\|$ and minimum γ values for the state feedback

State feedback	$\beta = 5$		$\beta = 10$		$\beta = 15$		$\beta = 20$	
factor × $a_s = 2.5$ $a_t = 1.5$	$\|F\| \times 10^3$	γ min $\times 10^8$	$\|F\| \times 10^3$	γ min $\times 10^8$	$\|F\| \times 10^3$	γ min $\times 10^8$	$\|F\| \times 10^3$	γ min $\times 10^8$
1	0·487	0·045	2·001	7·065	12·310	6·327	6·168	6·660
10	0·651	5·037	3·764	7·044	10·840	6·643	13·020	6·423
100	0·976	7·071	12·430	7·005	8·433	6·411	7·016	6·119
1000	2·014	7·071	12·880	6·657	10·820	6·124	35·410	6·492

Table II. $\|F\|$, $\|K\|$, and minimum γ values for the state-delayed feedback

State-delayed feedback	$\beta = 5$			$\beta = 10$			$\beta = 15$			$\beta = 20$		
factor × $a_s = 2{\cdot}5$ $a_t = 1{\cdot}5$	$\|F\| \times 10^3$	$\|K\| \times 10^3$	γ min $\times 10^4$	$\|F\| \times 10^3$	$\|K\| \times 10^3$	γ min $\times 10^4$	$\|F\| \times 10^3$	$\|K\| \times 10^3$	γ min $\times 10^4$	$\|F\| \times 10^3$	$\|K\| \times 10^3$	γ min $\times 10^4$
1	0·004	0·015	2·562	0·116	10·130	2·660	1·512	15·550	2·659	3·230	7·800	2·660
10	0·025	0·011	2·655	0·392	53·190	2·658	1·560	18·790	2·659	6·650	2·570	2·640
100	0·226	0·018	2·608	0·823	29·340	2·659	4·339	12·820	2·656	7·820	19·200	2·500
1000	0·629	0·017	2·659	3·005	50·660	2·658	13·580	12·390	2·612	7·960	6·720	2·630

Table III. $\|F\|$ and minimum γ values for the output feedback

Output feedback	$\beta = 5$		$\beta = 10$		$\beta = 15$		$\beta = 20$	
factor × $a_s = 2.5$ $a_t = 1.5$	$\|F\| \times 10^3$	γ min $\times 10^4$	$\|F\| \times 10^3$	γ min $\times 10^4$	$\|F\| \times 10^3$	γ min $\times 10^4$	$\|F\| \times 10^3$	γ min $\times 10^4$
1	0·549	0·302	1·877	2·659	14·110	2·511	14·150	2·513
10	0·651	2·634	3·708	2·662	10·410	2·478	6·491	2·462
100	0·975	2·659	9·950	2·616	7·079	2·429	7·086	2·508
1000	1·744	2·659	9·101	2·505	9·794	2·502	7·684	2·495

respectively. The obtained results support the potential use of β, a_s and a_t as tuning parameters in the process control design.

Computer simulation of the developed LMI-based control design methods have produced several feedback gains by tuning the parameters β, a_s and a_t. Appropriate values of the gain matrices of the three matrices are given by

State feedback method:

$$F = 10^3 \times [0{\cdot}0040\ 0{\cdot}0007\ 0{\cdot}0016\ 0{\cdot}0069\ \ -0{\cdot}0029\ \ -0{\cdot}0051\ 0{\cdot}7675 - 0{\cdot}0030\ 3{\cdot}0795\ 0{\cdot}0712]$$

State delayed feedback method:

$$F = [-0{\cdot}0189\ \ -0{\cdot}1931\ 0{\cdot}3479\ 0{\cdot}0507\ \ -0{\cdot}1646\ \ -0{\cdot}3318\ 54{\cdot}8052\ \ -4{\cdot}6952\ 74{\cdot}6564\ 4{\cdot}8511]$$

$$K = [0{\cdot}0008\ 0{\cdot}0008\ 0{\cdot}0011\ 0{\cdot}0011\ 0{\cdot}0004\ 0{\cdot}0006\ 0{\cdot}1978\ 0{\cdot}0328\ 0{\cdot}6714\ 0{\cdot}0363]$$

Static output feedback method:

$$F = 10^3 \times [0{\cdot}0045\ 0{\cdot}0014\ 0{\cdot}0009\ 0{\cdot}0078\ \ -0{\cdot}0030\ \ -0{\cdot}0051\ 0{\cdot}7017\ \ -0{\cdot}0026\ 3{\cdot}1615\ 0{\cdot}0644]$$

Two evaluation criteria were described in the benchmark definition: the first criterion is the 'RMS Responses' which is applied to the response of the structure when a Kanai–Tajimi filter is incorporated as input excitation. The second criterion is the 'Peak Responses' which is applied to the response of the structure when either of the two historical earthquakes (ElCentro and Hachinohe) are applied.

Indices $J_1 \rightarrow J_5$ are used to evaluate the 'RMS Responses' and $J_6 \rightarrow J_{10}$ for the 'Peak Responses'. Definitions of these indices were provided with the Benchmark Problem.[4]

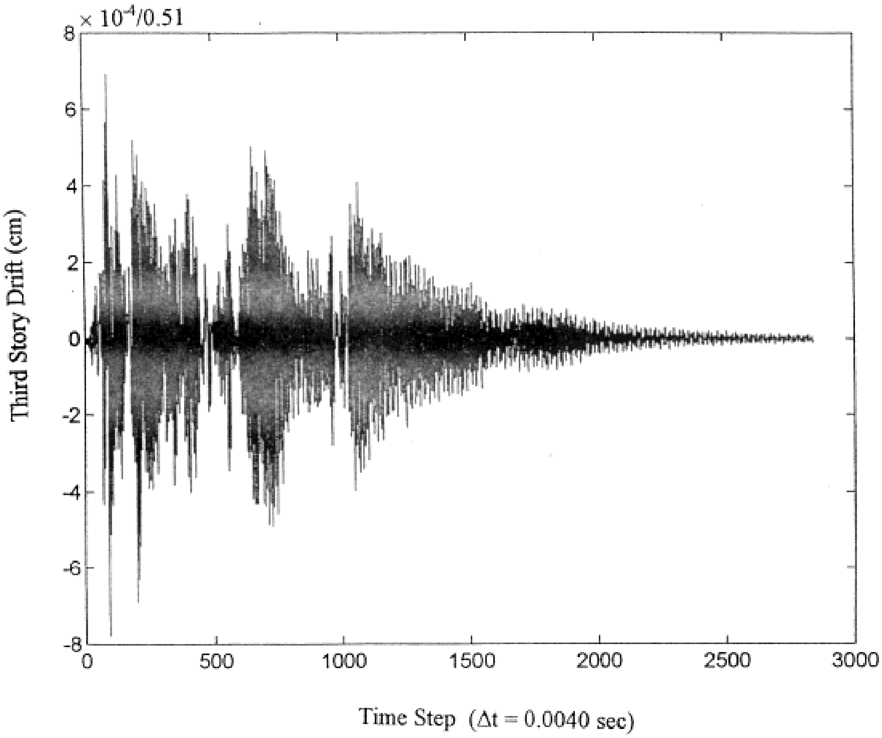

Figure 11. Sample of the vibration response of the drift of the third floor due to ElCentro earthquake

In order to evaluate the effect of time-delay per model (4) using the indices $J_1 \rightarrow J_5$ and $J_6 \rightarrow J_{10}$, the control gain matrices derived for the different values of β, a_s and a_t were utilized in the SIMULINK model of Fig. 2 with the three excitation inputs. The corresponding values of the indices $J_1 \rightarrow J_5$ and $J_6 \rightarrow J_{10}$ using our control design methods are presented in Tables IV and V, respectively. The data reported in Table IV represent an estimate of the worst case ω_g, ζ_g values for our control design. It is significant to record that the

Table IV. Largest evaluation indices for different ω_g & ζ_g values of the Kanai–Tajimi Spectrum

Kanai–Tajimi spectrum		$J_1 \times 10^{-3}$		$J_2 \times 10^{-3}$		$J_3 \times 10^{-3}$		$J_4 \times 10^{-3}$		$J_5 \times 10^{-3}$	
ω_g	ζ_g	120	0·30	37·3	0·30	120	0·60	37·3	0·30	37·3	0·30
J values		4·6000		0·2336		0·2509		0·1049		0·0011	

Table V. Evaluation indices for the ElCentro and Hachinohe earthquakes

	$J_6 \times 10^{-3}$	$J_7 \times 10^{-3}$	$J_8 \times 10^{-3}$	$J_9 \times 10^{-3}$	$J_{10} \times 10^{-3}$
Hachinohe Earthquake	2·9000	0·3376	0·4357	0·1798	0·1585
ElCentro Earthquake	2·3000	0·3803	0·4039	0·1755	0·0179

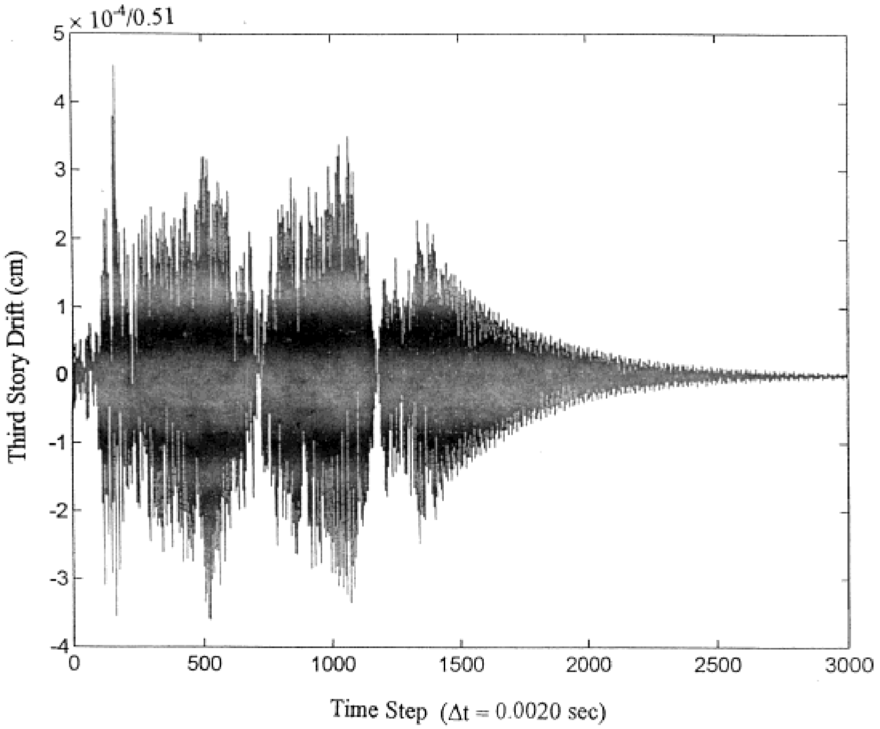

Figure 12. Sample of the vibration response of the drift of the third floor due to Hachinohe earthquake

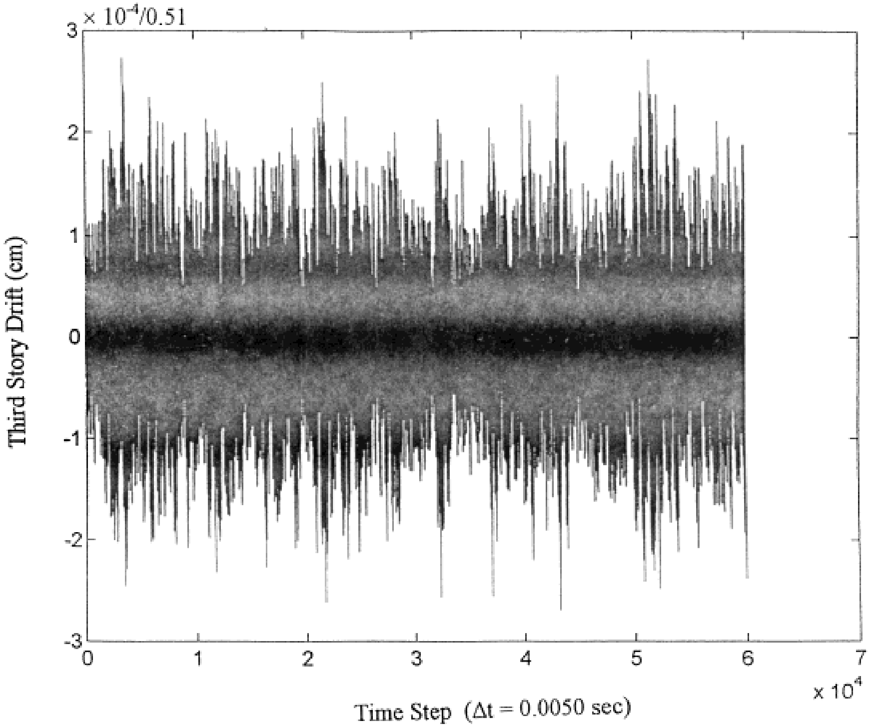

Figure 13. Sample of the vibration response of the drift of the third floor due to the Kanai–Tajimi Filter

Table VI. Evaluation indices at different values of ω_g and ζ_g for the Kanai–Tajimi spectrum

ζ_g	ω_g	$J_1 \times 10^{-5}$	$J_2 \times 10^{-5}$	$J_3 \times 10^{-5}$	$J_4 \times 10^{-5}$	$J_5 \times 10^{-5}$
	20	68·230	9·725	4·486	4·342	0·452
	37·3	110·000	23·355	6·598	10·486	1·095
0·30	60	190·000	13·937	9·494	6·260	0·660
	80	300·000	10·325	13·098	4·625	0·498
	100	440·000	8·733	17·811	3·886	0·434
	120	460·000	7·755	23·123	3·434	0·396
	20	100·000	11·895	6·879	5·325	0·558
	37·3	150·000	19·152	9·742	8·597	0·902
0·45	60	230·000	13·800	13·348	6·194	0·661
	80	310·000	10·711	16·928	4·797	0·527
	100	370·000	9·082	20·842	4·053	0·462
	120	380·000	8·046	24·821	3·583	0·425
	20	130·000	13·125	8·978	5·881	0·619
	37·3	190·000	16·610	12·458	7·454	0·789
0·60	60	260·000	12·910	16·365	5·791	0·628
	80	310·000	10·400	19·595	4·657	0·526
	100	330·000	8·899	22·568	3·977	0·471
	120	328·000	7·904	25·086	3·527	0·441
	20	159·000	13·662	10·753	6·124	0·648
	37·3	217·000	14·869	14·661	6·671	0·714
0·75	60	271·000	11·910	18·543	5·339	0·593
	80	296·000	9·850	21·219	4·410	0·515
	100	302·000	8·510	23·247	3·805	0·472
	120	292·000	7·587	24·605	3·390	0·449

values of these indices were not affected by the type of feedback control law or the values of β, a_s and a_t. Since $J_1 \rightarrow J_5$ depend on ω_g, ζ_g, Table VI displays representative values of such dependence.

Finally, samples of the vibration response of the drift of the third floor due to ElCentro earthquake, Hachinohe earthquake and the Kanai–Tajimi filter are presented in Figures 11–13, respectively. The vibration behaviour due to ElCentro and Hachinohe excitations were extended beyond the duration of the earthquakes (10 and 7.2 sec, respectively) to examine the decay of the response of the system; see Figures 11 and 12. This was difficult to achieve for the Kanai–Tajimi filter (whose duration is 300 sec) due to the nature of the equations controlling the excitation; see Figure 13.

10. CONCLUSIONS

This paper has been concerned with H_∞-feedback control design methods applied to a three-storey building with an active mass damper as a control mechanism. The design problem has been cast into the H_∞-control design of a class of dynamical systems with state and input delays. Time-delay factors arise mainly from two sources: The first stems from finite processing of information transfer due to the use of digital equipments. The second source is due to the limited nature of sensor data gathering and actuator signal processing. All the control design procedures have been expressed in the formalism of linear matrix inequalities (LMIs) and solved by the LMI-Control Toolbox. The developed controlled system is evaluated by computer simulation subjected to two historical earthquake excitation inputs (ElCentro and Hachinohe) and to the Kanai–Tajimi filter. This system has been successful in controlling the vibration response in the structure considered in this work. Typically, the drift of the third floor due to the various considered excitations demonstrate the effectiveness of this controlled system. Evaluation of the response of this system has been presented in the form of indices in order to compare with other solutions of the benchmark problem. Simulation results pertinent to the developed control design techniques are also presented. The obtained results have been found encouraging and provide useful guidelines for future designs.

REFERENCES

1. H. H. Leipholz (ed.), *Proc. 1st IUTAM Symp. on Structural Control*, North-Holland Publishers, Waterloo, 1979.
2. H. H. Leipholz (ed.), *Proc. 2nd IUTAM Symp. on Structural Control*, North-Holland Publishers, Waterloo, 1985.
3. Housner *et al*., *Proc.* 1*st World Conf. on Structural Control*, Los Angeles 1994.
4. B. F. Spencer *et al*., Benchmark problem in structural control, *Proc.* 1997 *ASCE Structures Congr*., Portland, Oregon, 13–16 April 1997.
5. S. J. Dyke *et al*., Absolute acceleration feedback control strategies for the active mass driver, *Proc.* 1 *st World Conf. on Structural Control*, Vol. 2, Pacadena, California, 3–5 August 1994, pp. TP1: 51–TP1: 60.
6. S. J. Dyke *et al*., Implementation of an active mass driver using acceleration feedback control, *Microcomput. Civil Engg*., in press.
7. B. A. Francis, *A Course in H_∞ Control Theory*, Springer, New York, 1987.
8. H. Kwakernaak, 'Robust control and H_∞-optimaization: tutorial paper', *Automatica* **29,** pp. 255–273 (1993).
9. G. Stepan, *Retarded Dynamical Systems*: *Stability and Characteristic Functions*, Longman Scientific & Technical, Essex, 1989.
10. M. Abdel-Rahman, Time-delay effects on actively damped structures, *J. Engng. Mech. ASCE*, **113,** 1709–1719 (1987).
11. H. S. Gorecki, P. Fuska, Grabowski and A. Korytowski, *Analysis and Synthesis of Time-Delay Systems*, Wiley, New York, 1989.
12. K. Zhou and P. P. Khargonekar, 'An algebraic Riccati equation approach to H_∞ optimization', *Systems Control Lett*. **11,** 85–91 (1988).
13. J. C. Doyle, K. Glover, P. P. Khargonekar and B. A. Francis, 'State-space solutions to standard H_2 and H_∞ control problems', *IEEE Trans. Automat. Control* **AC-34,** 831–847 (1989).
14. A. S. C. Sinha, 'Stabilization of large-scale nonlinear infinite delay systems', *Int. J. Systems Sci*., **21,** 2679–2684 (1990).
15. E. W. Kamen, P. P. Khargonekar and A. Tannenbaum, 'Dynamic control of systems with variable state-delay', *Acta Appl. Math.* **2,** 159–146 (1984).
16. D. O'Conner and T. J. Tarn, 'Dynamic control of systems with variable state', *IEEE Trans. Automat. Control*, **AC-28,** 123–146 (1983).
17. K. L. Cooke and J. M. Ferreira, 'Dynamic control of systems with variable state', *J. Math. Analysis Appl*., **96,** 480–490 (1983).
18. M. S. Mahmoud, 'H_∞ controllers for time-delay systems using LMI approach', *Technical Report ECE*-5448-97-01, Kuwait University, January 1997.
19. P. Gahinet, A. Nemirovski, A. J. Laub and M. Chilai, *LMI Control Toolbox*, The Mathworks, MA, 1995.
20. P. P. Khargonekar, I. R. Petersen and K. Zhou, 'Robust stabilization of uncertain linear systems: quadratic stabilizability and H_∞ control theory', *IEEE Trans. Automat. Control* **AC-35,** 356–361 (1990).

